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Efficiency and Solution Approaches to Bicriteria Nonconvex Programs

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Abstract. A new nonlinear scalarization specially designed for bicriteria nonconvex programming problems is presented. The scalarization is based on generalized Lagrangian duality theory and uses an augmented Lagrange function. The new concepts, q_i-approachable points and augmented duality gap, are introduced in order to determine the location of nondominated solutions with respect to a duality gap as well as the connectedness of the nondominated set.

Key words: Bicriteria, nonconvex, Lagrangian duality, nondominated solutions, q_i-approachable points, augmented duality gap.

1. Introduction

Theory and methodology of multiple objective programming (MOP) developed over the last twenty years deals primarily with linear or nonlinear problems satisfying some strong convexity assumptions. Applications in the field of engineering and management, however, may involve nonconvex objective functions defined over a nonconvex feasible set. Tabek *et al.* (1979) developed a nonconvex multiple criteria problem to model an aircraft control system; Ku Chi-Fa (1981) presented several applications of MOP modeling to chemical and mechanical engineering; Eschenauer *et al.* (1990) collected an impressive set of applied research projects on multiple criteria design optimization. In structural design, nonconvex sets and functions were used by Osyczka (1984) and Osyczka and Zajac (1990) to deal with a beam problem, a metal cutting problem, a machine tool gearbox problem, and the design of a robot spring balancing mechanism. Ashton and Atkins (1981) discussed ratios in multicriteria formulations in financial planning, and Lee and Wynne (1981) discussed ratios in multicriteria formulations in financial planning, and Lee and Wynne (1981) examined a goal programming model using nonlinear separable constraints. Recently, Kopsidas (1995) developed and studied a nonconvex bicriteria programming model for table olive preparations systems.

To solve a multiple objective problem generally means to find some or all of its efficient points and then find a 'best' solution among them according to the decision maker's preferences or overall utility. In the presence of various applications some research papers have been oriented toward generating efficient solutions of nonconvex MOP problems. Wendell and Lee (1977) proposed a scalarization combining the weighting method and the ε-constraint method, and used classical Lagrangian duality to show the existence of efficient points. In fact, Lagrangian duality theory has been related to MOP by many authors. For a summary, see Haimes and Chankong (1983). Wierzbicki's (1980, 1986) reference point methodology, which gave rise to many other research directions, uses scalarizing penalty functions with various types of norms in the objective space; some norms were specially designed for nonconvex problems. The augmented weighted Tchebycheff norm introduced by Steuer and Choo (1983) has attracted special interest due to its ability to deal with nonconvex problems. Using this norm, Kaliszewski (1986) obtained necessary and sufficient conditions for proper efficiency in MOP problems. Roy and Wallenius (1988) developed an interactive method suitable for handling nonlinear MOP problems including suggestions for nonconvex feasible regions and nonconvex objective functions. Kostreva *et al.* (1992) applied the scalarization technique of Benson (1978) and developed an approach for generating both locally and globally efficient solutions for polynomial (and thus, possibly highly nonconvex) MOP problems. Bernau (1990) applied exact penalty functions to determine efficient solutions. Studies on the connectedness of the efficient set and its image in the objective space, the nondominated set, were undertaken by Naccache (1978), Warburton (1983), Luc (1987), and very recently Hu and Sun (1993).

Much attention has been given to bicriteria programs (BCPs) due to their frequent occurrence in practice as well as their relative simplicity in comparison with multiple criteria problems. One of the first studies on generating efficient solutions of BCPs is due to Geoffrion (1967). Payne *et al.* (1975) characterized efficient solutions by means of a scalarizing function. Benson (1979) developed a parametric procedure for generating efficient solutions of convex BCPs. His procedure was based on the scalarization commonly referred to in the literature as the ε-constraint method and was later applied by Benson and Morin (1987) to a bicriteria nutrition planning problem. Gearhart (1979) also examined the ε-constraint scalarization, but for nonconvex problems. Bicriteria quasi-concave problems were studied by Schaible (1983) while bicriteria linear fractional programs were studied by Choo and Atkins (1982), Warburton (1985), Cambini and Martein (1988), and others. The structure of the nondominated set for BCPs ws analyzed by Martein (1988) and recently by TenHuisen and Wiecek (1995).

Interactive algorithms for finding the best efficient solution for BCPs have also been a subject of active studies. Among others, such algorithms were proposed by Walker (1978), Rietveld (1980), Payne and Polak (1980), and Aksoy (1990), who dealt with nonconvex mixed integer BCPs. Jahn and Merkel (1992), by means of the weighted Tchebycheff norm, developed an interactive approximation procedure for nonconvex BCPs and applied it to a mechanical engineering problem. Jahn *et al.* (1992) applied this procedure to design problems in chemical engineering, and Jueschke *et al.* (1995) applied it to antenna design. The paper of Payne *et al.* (1975) as well as the more recent papers of Payne (1993) and Helbig (1994) study approximation of the nondominated set of BCPs.

The research presented in this paper is aimed at contributing to the theory and methodology of bicriteria nonconvex programming problems. TenHuisen and Wiecek (1992, 1994) and TenHuisen (1993) developed foundations of a new approach for dealing specifically with nonconvex problems and for the first time related generalized Lagrangian duality to MOP. The generalized Lagrangian approach provides a basis for the development of new scalarization techniques which are capable of generating efficient solutions for problems whose nondominated points are located in the duality gap or whose nondominated set is disconnected.

In this paper, the BCP is related to its single objective counter-part for which two different generalized Lagrangian dual problems are developed. By making a specific selection as to the form of the generalized function, the new concepts of q_i-approachable points and augmented duality gap are introduced. Of particular interest is the ability of these dual problems to generate (locally) (weakly) efficient solutions, especially those which lie in a duality gap and are, therefore, inaccessible via other methods. The augmented duality gap helps determine the connectedness of the nondominated set. An extensive literature review conducted by the authors indicates that this generalized Lagrangian approach is the first method that generates an efficient point and informs about its location with regard to the structure of the nondominated set.

The paper is organized in the following manner. Section 2 includes problem formulation and terminology. Approachability and its relationship with efficiency are discussed in Section 3. Section 4 introduces a quadratic Lagrangian approach. Building upon the methodology of Section 4, the connectedness of the nondominated set is discussed in Section 5. Section 6 contains an illustrative example. Finally, conclusions on the results and some directions of further research are presented in Section 7.

2. Formulations and Definitions

The BCP considered in this paper is given as

$$\begin{aligned}\text{BCP:}\ \ &\text{minimize}\ \ \{f_1(x), f_2(x)\}\\ &\text{subject to}\ \ x \in X,\end{aligned}$$

where each $f_i(x), i = 1, 2,$ is a real-valued continuous function defined on $X \subseteq R^n$ and X is compact.

The concept of efficiency, first introduced by Pareto (1896), plays a central role in the theory presented in this paper. Three classifications of efficient solutions of BCP are defined here.

DEFINITION 2.1. A point $x^* \in X$ is called an *efficient solution of BCP* if there is no other $x \in X$ for which $f_i(x) \leq f_i(x^*)$ for both $i = 1, 2$, with strict inequality holding for at least one i.

DEFINITION 2.2. A point $x^* \in X$ is called a *weakly efficient solution of BCP* if there is no other $x \in X$ for which $f_i(x) < f_i(x^*)$ for both $i = 1, 2$.

DEFINITION 2.3. A point $x^* \in X$ is called a *properly efficient solution of BCP* if for each $i = 1, 2$, whenever $x \in X$ and $f_i(x) \leq f_i(x^*)$ it follows that for $j \neq i$,

i. $f_j(x) > f_j(x^*)$, and
ii. there exists a scalar $M > 0$ such that $(f_i(x) - f_i(x^*))/(f_j(x^*) - f_j(x)) \leq M$.

The sets of all efficient, weakly efficient, and properly efficient solutions are denoted here by X_E, X_{WE}, and X_{PE} respectively.

In addition to the classes of globally efficient solutions defined above, it is also possible to define corresponding classes of locally efficient solutions of BCP. A point $x^* \in X$ is said to be locally (weakly)(properly) efficient if there exists a neighborhood U of x^* such that x^* is (weakly) (properly) efficient over $U \cap X$. We denote the sets of locally efficient, locally weakly efficient, and locally properly efficient solutions by X_{LE}, X_{LWE}, and X_{LPE} respectively.

Many of the results contained in this paper are introduced and studied in the outcome or image space, R^2. Two sets which play fundamental roles in this study are the sets of outcomes for BCP, Y_1 and Y_2, which we define here as $Y_i := \{(y, z) = (f_j(x), f_i(x)), j \neq i : x \in X\}, i, j = 1, 2$. Note that these two sets are comprised of the images of exactly the same feasible points of BCP. However, the coordinates of every point in Y_1 are the reverse of the co-ordinates of that same point in Y_2. These different orderings of the co-ordinates of the points induce significant differences in the subsets and functions defined and examined in this paper.

Unless otherwise specified, the index i in the definitions and results contained in this paper is understood to be a given, fixed value of $i \in \{1, 2\}$ corresponding to the set Y_i. Such statements are true for both values of i independently.

The image of a (locally) (weakly) (properly) efficient solution under the vector-valued mapping $(f_j, f_i), i \neq j$, is called a (locally) (weakly) (properly) nondominated solution. The sets of nondominated, weakly nondominated, properly nondominated, locally nondominated, locally weakly nondominated, and locally properly nondominated solutions are each subsets of Y_i and are denoted here by $Y_{iE}, Y_{iWE}, Y_{iPE}, Y_{iLE}, Y_{iLWE}$, and Y_{iLPE} respectively.

REMARK 2.1. If $(y^*, z^*) \in Y_{iE}$, then $(z^*, y^*) \in Y_{jE}, j \neq i$. In addition, the same statement is true if the set Y_{iE} is replaced by any of the sets Y_{iWE}, Y_{iPE}, Y_{iLE}, Y_{iLWE}, or Y_{iLPE}.

Another subset of Y_i which we utilize throughout this paper is the set of lower envelope points (Martein, 1988), denoted here by Y_i^{env} and defined in Definition 2.4. Closely related to the set Y_i^{env} is the lower envelope function, $\text{env}_i(y)$, defined in Definition 2.5.

DEFINITION 2.4. A point $(y^*, z^*) \in Y_i$ is called a *lower envelope point of* Y_i if $z^* \leq z$ for all $(y, z) \in Y_i$ for which $y = y^*$.

The pre-image of a lower envelope point of Y_i in the decision space is called a pre-lower envelope point of Y_i. The set of all pre-lower envelope points of Y_i is denoted by X_i^{env}.

DEFINITION 2.5. Let $\text{env}_i(y)$:=min$\{z : (y, z) \in Y_i\}$.

From a comparison of the above definitions it can be seen that $(y, z) \in Y_i^{\text{env}}$ if and only if $\text{env}_i(y) = z$.

We now quote three theorems and a corollary, proven by TenHuisen and Wiecek (1995), which present relationships between points in X_i^{env} and various types of efficient solutions of BCP. These theorems are utilized in the proofs of theorems presented in Section 3 of this paper. The relationships presented are dependent upon the differentiability of the function $\text{env}_i(y)$. For the duration of this paper, $\text{env}_i'(y)$ and $\text{env}_i''(y)$ are used to denote the first and second derivatives of $\text{env}_i(y)$ respectively, and $\text{env}_i'(y^*) := \text{env}_i'(y)|_{y=y^*}$.

THEOREM 2.1. *A point $x^* \in X_E$ if and only if $x^* \in X_i^{\text{env}}$ for both $i = 1, 2$.*

THEOREM 2.2. *If $x^* \in X_{PE}$ and $\text{env}_i(y)$ is continuous and differentiable at $y^* = f_j(x^*)$, $j \neq i$, for both $i = 1, 2$, then $x^* \in X_i^{\text{env}}$ and $\text{env}_i'(y^*) < 0$ for both $i = 1, 2$.*

THEOREM 2.3. *Let $x^* \in X_i^{\text{env}}$ for some $i \in \{1, 2\}$. If $\text{env}_i(y)$ is continuous and differentiable at $y^* := f_j(x^*), j \neq i$, and $\text{env}_i'(y^*) < 0$ and finite, then $x^* \in X_{LPE}$.*

COROLLARY 2.1. *Let $x^* \in X_i^{\text{env}}$ for both $i = 1, 2$. If $\text{env}_i(y)$ is continuous and differentiable at $y^* = f_j(x^*), j \neq i$, and $\text{env}_i'(y^*) < 0$ and finite for both $i = 1, 2$, then $x^* \in X_{PE}$.*

The new Lagrangian dual approach presented in this paper generates solutions by means of a support function and uses the ε-constraint problem introduced by Benson and Morin (1977) and studied extensively by Haimes and Chankong (1983) and many others. This support function, however, is not linear and was designed to overcome the shortcomings encountered by the linear support function of the classical Lagrangian dual approach.

We first define what it means for a function to support a set in R^2.

DEFINITION 2.6. Let $(y^*, z^*) \in Y_i \subseteq R^2$. The function $f(y) : R \rightarrow R$ is said to support the set Y_i at (y^*, z^*) if $f(y^*) = z^*$ and $f(y) \leq z$ for all $(y, z) \in Y_i$.

Corresponding to BCP are the ε-constraint problems $P_i(\varepsilon), i = 1, 2$, given as

$$\begin{array}{lll} P_i(\varepsilon): & \text{minimize} & f_i(x) \\ & \text{subject to} & f_j(x) \leq \varepsilon, j \neq i \\ & & x \in X. \end{array}$$

It is assumed throughout this paper that the value of ε is always chosen such that $\min\{f_j(x) : x \in X\} \leq \varepsilon \leq \max\{f_j(x) : x \in X\}$.

Note that given ε we can find the value of $\text{env}_i(y)|_{y=\varepsilon}$ by solving $P_i(\varepsilon)$ with the constraint altered to be $f_j(x) = \varepsilon$.

Given $P_i(\varepsilon)$ as the primal problem, the classical Lagrangian dual problem, $LD_i(\varepsilon)$, is given as

$$\begin{array}{lll} LD_i(\varepsilon): & \text{maximize} & \Omega_i(\lambda, \varepsilon) \\ & \text{subject to} & \lambda \geq 0, \end{array}$$

where $\Omega_i(\lambda, \varepsilon) := \min\{L_i(x, \lambda, \varepsilon) : x \in X\}$, and $L_i(x, \lambda, \varepsilon) := f_i(x) + \lambda(f_j(x) - \varepsilon)$. This function $L_i(x, \lambda, \varepsilon)$ is referred to as the (classical) Lagrange function.

Although the ε-constraint method is effective at generating efficient solutions, it is unable to generate locally efficient solutions which are not also globally efficient. Furthermore, if Y_{iE} is disconnected (a property naturally occurring in nonconvex problems) then $P_i(\varepsilon)$ might also be unable to generate weakly efficient solutions. Its structure as a single objective optimization problem, however, makes it a powerful component in the development of the solution techniques that follow.

In general, the Lagrangian dual approach is even less effective than the ε-constraint method at generating (locally) (weakly) efficient solutions. For nonconvex problems, it is likely that $LD_i(\varepsilon)$ is incapable of generating all, or even most, of the efficient solutions. This problem cannot generate locally efficient solutions, and it might be unable to generate all weakly efficient solutions, regardless of whether Y_{iWE} is connected or not.

3. Approachability

In this section we introduce the concept of approachability and show how it relates to efficiency. We begin by proving results relating the function $\text{env}_i(y)$ to the quadratic support function $q_i(y) = -a(y - \varepsilon)^2 - b(y - \varepsilon) + c$, where a, b, and c are scalars.

In the results that follow, $q_i'(y)$ denotes the derivative of $q_i(y)$, and $q_i'(y^*) := q_i'(y)|_{y=y^*}$. Additionally, $V := \{y : y^* - \delta < y < y^* + \delta\}$, with $\delta > 0$, is used to denote a neighborhood of y^*, and $D_i := \{y = f_j(x), j \neq i : x \in X\}$ denotes the domain of the function $\text{env}_i(y)$.

LEMMA 3.1. *Let* $q_i(y) = -a(y-\varepsilon)^2 - b(y-\varepsilon) + c$. *If* $env_i(y)$ *is continuous and differentiable at* y^*, *then* $q_i(y^*) = env_i(y^*)$ *and* $q_i'(y^*) = env_i'(y^*)$ *if and only if for any value of* a, $b = -2a(y^*-\varepsilon) - env_i'(y^*)$ *and* $c = -a(y^*-\varepsilon)^2 - env_i'(y^*)(y^* - \varepsilon) + env_i(y^*)$.

Proof. The proof is straightforward and, therefore, omitted.

LEMMA 3.2. *Let* $(y^*, z^*) \in Y_i^{\text{env}}$ *and* $q_i(y) = -a(y-\varepsilon)^2 - b(y-\varepsilon) + c$, *with* $a > 0$. *If there exists a neighborhood* V *of* y^* *over which* $env_i(y)$ *is continuous and differentiable, and*

$$q_i(y^*) = env_i(y^*), \tag{1}$$

$$q_i'(y) \geq env_i'(y) \text{ for all } y \in V \text{ such that } y < y^*, \text{ and} \tag{2}$$

$$q_i'(y) \leq env_i'(y) \text{ for all } y \in V \text{ such that } y > y^*, \tag{3}$$

then $q_i(y) \leq env_i(y)$ *for all* $y \in V$.

Proof. (*by contradiction*). Assume that there exists $\hat{y} \in V$ such that

$$q_i(\hat{y}) > \text{env}_i(\hat{y}). \tag{4}$$

The proof is divided into cases which cover all possible orderings of the values $\hat{y}, y^*$, and $-b/(2a) + \varepsilon$ (the y-coordinate of the vertex of $q_i(y)$).

Case 1 $(-b/(2a) + \varepsilon \leq \hat{y} < y^*)$. Cauchy's Mean-Value Theorem (Gaughan, 1975) states that there exists a value $\bar{y}$ with $\hat{y} < \bar{y} < y^*$ such that $[q_i(y^*) - q_i(\hat{y})]\,\text{env}_i'(\bar{y}) = [\text{env}_i(y^*) - \text{env}_i(\hat{y})]q_i'(\bar{y})$. If (1) holds, then

$$[q_i(y^*) - q_i(\hat{y})]\,\text{env}_i'(\bar{y}) = [q_i(y^*) - \text{env}_i(\hat{y})]q_i'(\bar{y}). \tag{5}$$

Since $q_i(y)$ is a quadratic function which opens downward with the y-co-ordinate of its vertex at $-b/(za) + \epsilon$ and $-b/(za) + \epsilon \leq \hat{y} < \bar{y} < y^*$,

$$0 = q_i'(-b/(2a) + \varepsilon) \geq q_i'(\hat{y}) > q_i'(\bar{y}) > q_i'(y^*), \quad \text{and} \tag{6}$$

$$q_i(-b/(2a) + \varepsilon) \geq q_i(\hat{y}) > q_i(\bar{y}) > q_i(y^*). \tag{7}$$

From (4) and (6) it follows that

$$[q_i(y^*) - \text{env}_i(\hat{y})]q_i'(\bar{y}) < [q_i(y^*) - q_i(\hat{y})]q_i'(\bar{y}). \tag{8}$$

Combining (5) with (8) results in

$$[q_i(y^*) - q_i(\hat{y})]\text{env}_i'(\bar{y}) < [q_i(y^*) - q_i(\hat{y})]q_i'(\bar{y}). \tag{9}$$

Inequalities (7) and (9) imply that $\text{env}_i'(\bar{y}) > q_i'(\bar{y})$. Since $\bar{y} < y^*$, this contradicts inequality (2). Therefore, $q_i(y) \leq \text{env}_i(y)$ for all $y \in V$ such that $-b/(2a) + \varepsilon \leq y < y^*$.

Case 2 $(\hat{y} < -b/(2a) + \varepsilon \leq y^*)$. From Cauchy's Mean-Value Theorem, there exists a value $\bar{y}$ with $\hat{y} < \bar{y} < -b/(2a) + \varepsilon$ such that

$$[q_i(-b/(2a) + \varepsilon) - q_i(\hat{y})]\text{env}_i'(\hat{y}) \\ = [\text{env}_i(-b/(2a) + \varepsilon) - \text{env}_i(\hat{y})]q_i'(\bar{y}). \tag{10}$$

From the structure of $q_i(y)$ and the fact that $\hat{y} < \bar{y} < -b/(2a) + \varepsilon \leq y^*$,

$$q_i'(\hat{y}) > q_i'(\bar{y}) > q_i'(-b/(2a) + \varepsilon) = 0 \geq q_i'(y^*), \quad \text{and} \tag{11}$$

$$q_i(\hat{y}) < q_i(\bar{y}) < q_i(-b/(2a) + \varepsilon) \geq q_i(y^*). \tag{12}$$

From (4) and the fact that $q_i(-b/(2a) + \varepsilon) \leq \text{env}_i(-b/(2a) + \varepsilon)$ (established in case 1),

$$q_i(-b/(2a) + \varepsilon) - q_i(\hat{y}) < \text{env}_i(-b/(2a) + \varepsilon) - \text{env}_i(\hat{y}). \tag{13}$$

Inequalities (12) and (13) imply that

$$q_i(-b/(2a) + \varepsilon) - q_i(\hat{y}) > 0 \quad \text{and} \tag{14}$$

$$\text{env}_i(-b/(2a) + \varepsilon) - \text{env}_i(\hat{y}) > 0. \tag{15}$$

It follows from (10), (11), (13), (14), and (15) that $\text{env}_i'(\bar{y}) > q_i'(\bar{y})$. Since $\bar{y} < y^*$, this contradicts inequality (2). Therefore, $q_i(y) \leq \text{env}_i(y)$ for all $y \in V$ such that $y < -b/(2a) + \varepsilon \leq y^*$.

The proofs of the three cases with $-b/(2a)+\varepsilon \leq y^* < \hat{y}, \hat{y} < y^* \leq -b/(2a)+\varepsilon$, and $y^* < \hat{y} \leq -b/(2a) + \varepsilon$ follow closely the proof of case 1, and the proof of the case with $y^* \leq -b/(2a) + \varepsilon < \hat{y}$ is nearly identical to that of case 2. The compilation of the six cases proves that $q_i(y) \leq \text{env}_i(y)$ for all $y \in V$. ■

LEMMA 3.3. *Let* $(y^*, z^*) \in Y_i^{\text{env}}$ *and* $q_i(y) = -a(y - \varepsilon)^2 - b(y - \varepsilon) + c$, *with* $a > 0$ *(and finite). If there exists a neighborhood* V *of* y^* *over which* env$_i(y)$ *is continuous and differentiable, and*

$$q_i(y^*) = \mathit{env}_i(y^*), \tag{16}$$

$$q_i'(y^*) = \mathit{env}_i'(y^*), \quad \textit{and} \tag{17}$$

$$q_i(y) \leq \mathit{env}_i(y) \textit{ for all } y \in V, \tag{18}$$

then there exists a (finite) scalar $\bar{a} > 0$ *such that* $q_i(y)$ *supports* Y_i *at* (y^*, z^*) *for all* $a \geq \bar{a}$.

Proof. From (16) and (17), Lemma 3.1 implies that $b = -2a(y^* - \varepsilon) - \text{env}_i'(y^*)$ and $c = -a(y^* - \varepsilon)^2 - \text{env}_i'(y^*)(y^* - \varepsilon) + \text{env}_i(y^*)$. Consequently,

$$q_i(y) = -a(y - y^*)^2 + \text{env}_i'(y^*)(y - y^*) + \text{env}_i(y^*). \tag{19}$$

Since $f_i(x)$ is continuous over X and X is compact, env$_i(y)$ achieves a finite minimum over D_i. Moreover, if the value of $a > 0$ is increased, it follows from (19) that $q_i(y)$ will strictly decrease for all $y \neq y^*$. Therefore, it is possible to find a (finite) scalar $\bar{a} > 0$ such that

$$\max\{q_i(y^* - \delta), q_i(y^* + \delta)\} < \min\{\text{env}_i(y) : y \in D_i\}. \tag{20}$$

That is, for some (finite) value $\bar{a} > 0$ the value of $q_i(y)$ at each endpoint of the interval defining V is less than the minimum value of env$_i(y)$ over its entire domain. Also, (19) implies that

$$q_i(y) < q_i(y^* - \delta) \text{ for all } y < y^* - \delta, \text{ and} \tag{21}$$

$$q_i(y) < q_i(y^* + \delta) \text{ for all } y > y^* + \delta, \tag{22}$$

where $\delta > 0$. From (18), (20), (21), and (22) it follows that there exists a (finite) scalar $\bar{a} > 0$ such that $q_i(y) \leq \text{env}_i(y)$ for all $y \in D_i$ and $a \geq \bar{a}$. Combining this with (16) completes the proof. ■

COROLLARY 3.1. *Let $(y^*, z^*) \in Y_i^{\text{env}}$ and $q_i(y) = -a(y - \varepsilon)^2 - b(y - \varepsilon) + c$, with $a > 0$ (and finite). If there exists a neighborhood V of y^* over which env$_i(y)$ is continuous and differentiable, and*

$$q_i(y^*) = \mathit{env}_i(y^*), \tag{23}$$

$$q_i'(y) \geq \mathit{env}_i'(y) \textit{ for all } y \in V \textit{ such that } y < y^*, \textit{ and} \tag{24}$$

$$q_i'(y) \leq \mathit{env}_i'(y) \textit{ for all } y \in V \textit{ such that } y > y^*, \tag{25}$$

then there exists a (finite) scalar $\bar{a} > 0$ such that $q_i(y)$ supports Y_i at (y^*, z^*) for all $a \geq \bar{a}$.

Proof. From (23), (24), and (25), Lemma 3.2 implies that

$$q_i(y) \leq \text{env}_i(y) \text{ for all } y \in V. \tag{26}$$

Inequalities (24), (25), and the fact that env$_i(y)$ is differentiable over V imply that

$$q_i'(y^*) = \text{env}_i'(y^*). \tag{27}$$

Applying (23), (26), and (27) to Lemma 3.3 completes the proof. ■

We now introduce the concept of approachability. The relationship which approachability has with lower envelope points and nondominated solutions of BCP (and the pre-images of such points in the decision space) is presented in the theorems that follow in this section.

DEFINITION 3.1. A point $x^* \in X$ is called *q_i-approachable* if there exists a quadratic function of the form $q_i(y) = -a(y-\varepsilon)^2 - b(y-\varepsilon) + c$, with $a > 0$, which supports the set Y_i at the point (y^*, z^*), where $z^* := f_i(x^*)$ and $y^* := f_j(x^*), j \neq i$.

Although not explicitly stated in the above definition, it is permissible for the coefficient $a > 0$ in $q_i(y)$ to be infinite. However, we are primarily interested in the cases for which it is possible to find a finite value of $a > 0$ such that the quadratic function $q_i(y)$ supports the set Y_i. Under the conditions of continuity and differentiability, the cases which would require an infinite value of $a > 0$ are extremely rare.

THEOREM 3.1. *If there exists $x^* \in X_i^{\text{env}}$ and a neighborhood V of $y^* := f_j(x^*), j \neq i$, such that env$_i(y)$ is continuous and differentiable over V, then there exists a scalar $\bar{a} > 0$ such that x^* is q_i-approachable for any $a \geq \bar{a}$.*

Proof. Let $q_i(y) = -a(y-\varepsilon)^2 - b(y-\varepsilon) + c$, with $a > 0, b = -2a(y^* - \varepsilon) - \text{env}_i'(y^*)$, and $c = -a(y^* - \varepsilon)^2 - \text{env}_i'(y^*)(y^* - \varepsilon) + \text{env}_i(y^*)$. Then Lemma 3.1 proves that

$$q_i(y^*) = \text{env}_i(y^*), \tag{28}$$

for any value of $a > 0$. If $\hat{a} > a > 0$, then $-2\hat{a}(y - y^*) > -2a(y - y^*)$ if $y < y^*$ and $-2\hat{a}(y - y^*) < -2a(y - y^*)$ if $y > y^*$. Therefore, since $q_i'(y) = -2a(y - y^*) + \text{env}_i'(y^*)$, increasing the value of $a > 0$ increases the value of $q_i'(y)$ if $y < y^*$ and decreases the value of $q_i'(y)$ if $y > y^*$. Thus, there exists a value $a^* > 0$ (possibly infinite) such that

$$q_i'(y) \geq \text{env}_i'(y) \text{ for all } y \in V \text{ such that } y < y^*, \text{ and} \tag{29}$$

$$q_i'(y) \leq \text{env}_i'(y) \text{ for all } y \in V \text{ such that } y > y^*. \tag{30}$$

By Corollary 3.1, (28), (29), and (30) imply that there exists a scalar $\bar{a} \geq a^* > 0$ such that $q_i(y)$ supports Y_i at (y^*, z^*) for any value of $a \geq \bar{a}$. ∎

THEOREM 3.2. *If there exist $x^* \in X_i^{\text{env}}$ and a neighborhood V of $y^* := f_j(x^*), j \neq i$, such that env$_i(y)$ is continuous and twice differentiable over V with env$_i'(y)$ finite and env$_i''(y) > -\infty$ over V, then there exists a finite value of $a^* > 0$ such that x^* is q_i-approachable for any $a \geq a^*$.*

Proof. Let y be any point in V such that $y \neq y^*$, and define

$$a(y) := [\text{env}_i'(y) - \text{env}_i'(y^*)]/[-2(y - y^*)],$$

$$a_<^* := \sup\{1, \sup\{a(y) : y \in V, y < y^*\}\}, \text{ and}$$

$$a_>^* := \sup\{1, \sup\{a(y) : y \in V, y > y^*\}\}.$$

Since env$_i'(y)$ is finite over V, $a(y)$ is clearly finite for all $y \in V$ such that $y \neq y^*$. In addition, since it is assumed that env$_i''(y) > -\infty$ over V, $a(y)$ is finite for all

$y \in V$. Consequently, both $a^*_<$ and $a^*_>$ are finite. Let $a^* := \max\{a^*_<, a^*_>\}$ and $q_i(y) = -a^*(y-\varepsilon)^2 - b(y-\varepsilon) + c$, with $b = -2a^*(y^*-\varepsilon) - \text{env}_i'(y^*)$ and $c = -a^*(y^*-\varepsilon)^2 - \text{env}_i'(y^*)(y^*-\varepsilon) + \text{env}_i(y^*)$. Then conditions (23), (24), and (25) of Corollary 3.1 are satisfied with this finite scalar a^*. Therefore, there exists a finite scalar $\bar{a} \geq a^* > 0$ such that $q_i(y)$ supports Y_i at (y^*, z^*) for all $a \geq \bar{a}$. ■

Since Theorem 3.2 imposes stronger conditions on $\text{env}_i(y)$ than Theorem 3.1 in order to guarantee approachability with a finite coefficient a, we will illustrate their necessity with a small example. Consider $\text{env}_i(y) = -y^{1.5}$ and observe that this function cannot be supported at $y^* = 0$ by any quadratic function $q_i(y)$ with a finite leading coefficient.

THEOREM 3.3. *If $x^* \in X$ is q_i-approachable, then $x^* \in X_i^{\text{env}}$.*

Proof. If x^* is q_i-approachable, then there exists a quadratic function of the form $q_i(y) = -a(y-\varepsilon)^2 - b(y-\varepsilon) + c$, with $a > 0$, which supports Y_i at (y^*, z^*), where $z^* := f_i(x^*)$ and $y^* := f_j(x^*), j \neq i$. That is, $q_i(y^*) = z^*$ and $q_i(y) \leq z$ for all $(y, z) \in Y_i$. Therefore, there does not exist $\hat{x} \in X$ such that $\hat{y} := f_j(\hat{x}) = y^*$ and $\hat{z} := f_i(\hat{x}) < z^*$. It follows that $x^* \in X_i^{\text{env}}$. ■

THEOREM 3.4. *If, for some $i \in \{1, 2\}$, there exist $x^* \in X$ and a neighborhood V of $y^* := f_j(x^*), j \neq i$, such that x^* is q_i-approachable, $\text{env}_i(y)$ is continuous and differentiable over V, and $\text{env}_i'(y^*) < 0$, then $x^* \in X_{LPE}$.*

Proof. If $x^* \in X$ is q_i-approachable, Theorem 3.3 proves that $x^* \in X_i^{\text{env}}$. Theorem 2.3, then, implies that $x^* \in X_{LPE}$. ■

THEOREM 3.5. *If, for both $i = 1, 2$, there exist $x^* \in X$ and a neighborhood V of $y^* := f_j(x^*), j \neq i$, such that x^* is q_i-approachable, $\text{env}_i(y)$ is continuous and differentiable over V, and $\text{env}_i'(y^*) < 0$ and finite, then $x^* \in X_{PE}$.*

Proof. If, for both $i = 1, 2, x^*$ is q_i-approachable, Theorem 3.3 proves that $x^* \in X_i^{\text{env}}$, and since $\text{env}_i'(y^*) < 0$ and finite, Corollary 2.1 proves that $x^* \in X_{PE}$. ■

THEOREM 3.6. *If, for both $i = 1, 2$, there exist $x^* \in X$ and a neighborhood V of $y^* := f_j(x^*), j \neq i$, such that $\text{env}_i(y)$ is continuous, and differentiable over V, and $x^* \in X_{PE}$, then x^* is q_i-approachable for both $i = 1, 2$.*

Proof. If $x^* \in X_{PE}$, Theorem 2.2 proves that $x^* \in X_i^{\text{env}}$ for both $i = 1, 2$. In turn, Theorem 3.1 proves that x^* is q_i-approachable for both $i = 1, 2$. ■

In this section the concepts of the lower envelope function $\text{env}_i(y)$ and its derivative were used extensively. It was assumed throughout that $\text{env}_i(y)$ is continuous and differentiable over some neighborhood V of y^*. We point out here that in such cases one can easily approximate $\text{env}_i'(y^*)$ with the quantity $(z^* - z)/(y^* - y)$ for $(y, z) \neq (y^*, z^*)$ in $Y_i^{\text{env}} \cap V$.

The generalization of the differentiability requirement to subdifferentiability or B-differentiability would not resolve, in many instances, the lack of approachability at a given point. For example, $\mathrm{env}_i(y) = -|y|$ is both subdifferentiable and B-differentiable at $y = 0$, but it is q_i-approachable at this point only with a quadratic function $q_i(y)$ whose leading coefficient a is infinite. Thus, the approachability of such points would require a degenerate quadratic support function, which in effect would render the quadratic Lagrangian approach introduced and studied in this paper powerless.

4. A Quadratic Lagrangian Approach

In an attempt to resolve the duality gap that exists between nonlinear single objective primal problems and their corresponding dual problem, Bazarra (1973), Gould (1969), and Nakayama *et al*. (1975) examined the use of generalized Lagrangian duality as an alternative to the classical Lagrangian dual approach. TenHuisen (1993) and TenHuisen and Wiecek (1994) presented an extensive study of the application of generalized Lagrangian duality to multiple objective nonlinear programming problems.

In this section we examine the augmented Lagrange function, related to $P_i(\varepsilon)$, given by

$$QL_i(x, a, b, \varepsilon) := f_i(x) + a(f_j(x) - \varepsilon)^2 + b(f_j(x) - \varepsilon), \tag{31}$$

where a and b are the Lagrange multipliers. Rockafellar (1974) showed that this quadratic Lagrange function, when applied to the single objective nonlinear programming problem with equality constraints, may eliminate a duality gap. In his development, this function has to be modified when applied to the nonlinear program with inequality constraints. Various augmented Lagrangians have been studied by many authors. For excellent reviews see Tind and Wolsey (1981) and Minoux (1986).

The novelty of our approach is the application of the augmented Lagrange function given in (31) to $P_i(\varepsilon)$ even with the inequality constraint $f_j(x) \leq \varepsilon$ present in this nonlinear problem. Unlike Rockafellar, our goal is not eliminating a duality gap, but rather investigating its existence and the connectedness of the (weakly) nondominated set.

The formulation of the quadratic Lagrangian dual problem, $\mathrm{QLD}_i(\varepsilon)$, corresponding to (31) is given as

$$\begin{array}{lll} \mathrm{QLD}_i(\varepsilon): & \text{maximize} & \Theta_i(a, b, \varepsilon) \\ & \text{subject to} & a > 0 \\ & & b \text{ free}, \end{array}$$

where

$$\Theta_i(a, b, \varepsilon) := \min\{QL_i(x, a, b, \varepsilon) : x \in X\}. \tag{32}$$

Solving $\text{QLD}_i(\varepsilon)$ is equivalent to finding the quadratic function supporting the set Y_i having the greatest intercept with the line $y = \varepsilon$.

LEMMA 4.1. *Let* $q_i(y) = -a(y-\varepsilon)^2 - b(y-\varepsilon) + QL_i(x^*, a, b, \varepsilon)$. *Then* $x^* \in X$ *minimizes* $QL_i(x, a, b, \varepsilon)$ *over* X *if and only if* $q_i(y) \leq z$ *for all* $(y, z) \in Y_i$.

Proof. x^* minimizes $QL_i(x, a, b, \varepsilon)$ over X if and only if $QL_i(x^*, a, b, \varepsilon) \leq QL_i(x, a, b, \varepsilon)$ for all $x \in X$, which is equivalent to $QL_i(x^*, a, b, \varepsilon) \leq f_i(x) + a(f_j(x) - \varepsilon)^2 + b(f_j(x) - \varepsilon)$ for all $x \in X$. Letting $y := f_j(x)$ and $z := f_i(x)$, this becomes $QL_i(x^*, a, b, \varepsilon) \leq z + a(y-\varepsilon)^2 + b(y-\varepsilon)$ for all $(y, z) \in Y_i$. Hence $-a(y-\varepsilon)^2 - b(y-\varepsilon) + QL_i(x^*, a, b, \varepsilon) \leq z$ for all $(y, z) \in Y_i$. Finally, applying the definition of $q_i(y)$, we get that $q_i(y) \leq z$ for all $(y, z) \in Y_i$. ■

THEOREM 4.1. *The point* $x^* \in X$ *minimizes* $QL_i(x, a, b, \varepsilon)$ *over* X *if and only if* x^* *is* q_i*-approachable.*

Proof. $(\Rightarrow)$ Let x^* minimize $QL_i(x, a, b, \varepsilon)$ over X. Lemma 4.1, then, proves that

$$q_i(y) \leq z \text{ for all } (y, z) \in Y_i. \tag{33}$$

Now consider the quadratic function $q_i(y) = -a(y-\varepsilon)^2 - b(y-\varepsilon) + QL_i(x^*, a, b, \varepsilon)$, with $a > 0$. We shall show that this function $q_i(y)$ supports the set Y_i at (y^*, z^*), where $z^* := f_i(x^*)$ and $y^* := f_j(x^*), j \neq i$.

It follows from the definition of $q_i(y)$ and $QL_i(x^*, a, b, \varepsilon)$ that

$$\begin{aligned} q_i(y^*) &= -a(y^*-\varepsilon)^2 - b(y^*-\varepsilon) + QL_i(x^*, a, b, \varepsilon) \\ &= -a(y^*-\varepsilon)^2 - b(y^*-\varepsilon) + f_i(x^*) \\ &\quad + a(f_j(x^*) - \varepsilon)^2 + b(f_j(x^*) - \varepsilon) \\ &= -a(y^*-\varepsilon)^2 - b(y^*-\varepsilon) + z^* + a(y^*-\varepsilon)^2 + b(y^*-\varepsilon) \\ &= z^*. \end{aligned} \tag{34}$$

Inequality (33) and Equation (34) satisfy the conditions given in Definition 2.6 for the quadratic function $q_i(y) = -a(y-\varepsilon)^2 - b(y-\varepsilon) + QL_i(x^*, a, b, \varepsilon)$ to support Y_i at (y^*, z^*). Therefore, x^* is q_i-approachable.

$(\Leftarrow)$ Since x^* is q_i-approachable, there exists a quadratic function of the form $q_i(y) = -a(y-\varepsilon)^2 - b(y-\varepsilon) + c$, with $a > 0$, which supports the set Y_i at $(y^*, z^*) := (f_j(x^*), f_i(x^*))$. That is,

$$q_i(y) \leq z \text{ for all } (y, z) \in Y_i, \text{ and} \tag{35}$$

$$q_i(y^*) = z^*. \tag{36}$$

It follows from (36) and the formulation of $q_i(y)$ that $-a(y^*-\varepsilon)^2 - b(y^*-\varepsilon) + c = z^*$. Therefore,

$$-a(y^*-\varepsilon)^2 - b(y^*-\varepsilon) + c + QL_i(x^*, a, b, \varepsilon) = z^* + QL_i(x^*, a, b, \varepsilon).$$

Applying (31) gives

$$-a(y^*-\varepsilon)^2 - b(y^*-\varepsilon) + c + f_i(x^*) + a(f_j(x^*)-\varepsilon)^2 + b(f_j(x^*)-\varepsilon)$$
$$= z^* + QL_i(x^*, a, b, \varepsilon).$$

Since $z^* = f_i(x^*)$ and $y^* = f_j(x^*), j \neq i$, we get

$$-a(y^*-\varepsilon)^2 - b(y^*-\varepsilon) + c + z^* + a(y^*-\varepsilon)^2 + b(y^*-\varepsilon)$$
$$= z^* + QL_i(x^*, a, b, \varepsilon),$$

which yields $c = QL_i(x^*, a, b, \varepsilon)$. Therefore,

$$q_i(y) = -a(y-\varepsilon)^2 - b(y-\varepsilon) + QL_i(x^*, a, b, \varepsilon). \tag{37}$$

From (35) and (37), Lemma 4.1 proves that x^* minimizes $QL_i(x, a, b, \varepsilon)$ over X. ■

The preceding theorem shows that if the point x^* is q_i-approachable, then it minimizes $QL_i(x, a, b, \varepsilon)$ over X. It also proves that minimizing the Lagrangian function $QL_i(x, a, b, \varepsilon)$ with fixed values of $a > 0$ and b produces a q_i-approachable point. Theorem 4.2 below provides a comparison of the optimal values of $QLD_i(\varepsilon)$ and $P_i(\varepsilon)$.

THEOREM 4.2. *Suppose that there exists a neighborhood V of $y = \varepsilon$ such that env$_i(y)$ is continuous and differentiable over V. If x^* solves $P_i(\varepsilon)$ and (a^*, b^*) solves $QLD_i(\varepsilon)$, then $f_i(x^*) \leq \Theta_i(a^*, b^*, \varepsilon)$.*

Proof. Let $\bar{x} \in X_i^{\text{env}}$ for some $i \in \{1, 2\}$, such that $f_j(\bar{x}) = \varepsilon, j \neq i$. Theorem 3.1 proves that $\bar{x}$ is q_i-approachable. In turn, Theorem 4.1 implies that $\bar{x}$ minimizes $QL_i(x, a, b, \varepsilon)$ over X. Then, from (32),

$$QL_i(\bar{x}, a, b, \varepsilon) = \Theta_i(a, b, \varepsilon). \tag{38}$$

Moreover, from (31) it follows that

$$QL_i(\bar{x}, a, b, \varepsilon) = f_i(\bar{x}) + a(f_j(\bar{x}) - \varepsilon)^2 + b(f_j(\bar{x} - \varepsilon)). \tag{39}$$

Since $f_j(\bar{x}) = \varepsilon$, equation (39) becomes

$$QL_i(\bar{x}, a, b, \varepsilon) = f_i(\bar{x}). \tag{40}$$

Equations (38) and (40) imply that

$$\Theta_i(a, b, \varepsilon) = f_i(\bar{x}). \tag{41}$$

Since (a^*, b^*) solves $QLD_i(\varepsilon)$,

$$\Theta_i(a^*, b^*, \varepsilon) \geq \Theta_i(a, b, \varepsilon) \text{ for all } a > 0. \tag{42}$$

Since x^* solves $P_i(\varepsilon)$, $f_i(x^*) \leq f_i(x)$ for all $x \in X$ such that $f_j(x) \leq \varepsilon$. Since $f_j(\bar{x}) = \varepsilon$,

$$f_i(x^*) \leq f_i(\bar{x}). \tag{43}$$

Combining (41), (42), and (43) completes the proof. ■

Theorem 4.2 proves that in general the optimal value of the primal problem $P_i(\varepsilon)$ is less than or equal to the optimal value of the dual problem $QLD_i(\varepsilon)$. Theorem 4.3 and Corollary 4.1 below specify optimal values of the dual problem.

THEOREM 4.3. *If x^* is q_i-approachable and $f_j(x^*) = \varepsilon$, then $\Theta_i(a^*, b^*, \varepsilon) = f_i(x^*)$, where (a^*, b^*) is a solution of $QLD_i(\varepsilon)$.*

Proof. Since x^* is q_i-approachable, Theorem 4.1 proves that x^* minimizes $QL_i(x, a^*, b^*, \varepsilon)$ over X. From (32), then

$$QL_i(x^*, a^*, b^*, \varepsilon) = \Theta_i(a^*, b^*, \varepsilon). \tag{44}$$

From (31) and the fact that $f_j(x^*) = \varepsilon$, it follows that

$$QL_i(x^*, a, b, \varepsilon) = f_i(x^*). \tag{45}$$

Combining (44) and (45) completes the proof. ■

COROLLARY 4.1. *Suppose there exists $x^* \in X_i^{\text{env}}$ and a neighborhood V of $y^* := f_j(x^*) = \varepsilon$ such that $\text{env}_i(y)$ is continuous and differentiable over V. If (a^*, b^*) solves $QLD_i(\varepsilon)$, then $f_i(x^*) = \Theta_i(a^*, b^*, \varepsilon)$.*

Proof. Since $x^* \in X_i^{\text{env}}$, Theorem 3.1 implies that x^* is q_i-approachable. Since $f_j(x^*) = \varepsilon$, Theorem 4.3 states that $f_i(x^*) = \Omega_i(a^*, b^*, \varepsilon)$. ■

5. Augmented Duality Gap

A duality gap is said to exist between the primal problem, $P_i(\varepsilon)$, and its corresponding dual problem formulated with the classical linear Lagrange function, $LD_i(\varepsilon)$, if the optimal value of the primal problem is strictly greater than the optimal value of the dual problem. Even though such a gap might exist, it is likely that points in this gap are q_i-approachable. It is of interest to us to examine the properties of those q_i-approachable points which lie in the duality gap. In particular, we are interested in determining whether any of these points are (weakly) efficient solutions of BCP. In order to make this assessment, we first introduce a means of categorizing duality gaps with Definition 5.1 and define what it means for the set Y_{iWE} to be disconnected in Definition 5.2.

DEFINITION 5.1. A duality gap which exists between the primal problem $P_i(\varepsilon)$ and its classical dual problem $LD_i(\varepsilon)$ is said to be *augmented* if there exist $x_1, x_2 \in X_i^{\text{env}}$ such that $f_i(x_1) < f_i(x_2)$ and $f_j(x_1) < f_j(x_2) \leq \varepsilon, j \neq i$.

DEFINITION 5.2. The set Y_{iWE} is disconnected if there exist $x_1, x_2 \in X_{WE}$ and $\bar{x} \notin X_{WE}$ such that $\bar{x} \in X_i^{\text{env}}$ and $f_j(x_1) < f_j(\bar{x}) < f_j(x_2)$.

The existence of an augmented duality gap and the disconnectedness of the set of weakly nondominated solutions are closely related. In this section we explore this relationship and how these two concepts affect the solution of $QLD_i(\varepsilon)$.

For ease of notation, let

$$x^i := \text{arg min}\{f_j(x), j \neq i : x \in \{\text{arg min}\{f_i(x) : x \in X\}\}\}. \tag{46}$$

THEOREM 5.1. *Y_{iWE} is connected if and only if there is no augmented duality gap for any value of $\varepsilon \leq y^i$, where $y^i = f_j(x^i), j \neq i$, and x^i is defined as in (46).*

Proof. (by contradiction). $(\Rightarrow)$ Assume that there exists $x_0 \in X$ such that an augmented duality gap exists for $\varepsilon = y_0 := f_j(x_0), j \neq i$, with

$$f_j(x_0) \leq f_j(x^i). \tag{47}$$

Definition 5.1 states that there exist

$$x_1, x_2 \in X_i^{\text{env}} \tag{48}$$

such that

$$f_j(x_1) < f_j(x_2) \leq f_j(x_0), \text{ and} \tag{49}$$

$$f_i(x_1) < f_i(x_2), \tag{50}$$

and consequently

$$x_2 \notin X_{WE}. \tag{51}$$

Furthermore, from the definition of x^j in (46) it follows that

$$f_j(x^j) \leq f_j(x_1). \tag{52}$$

Also, since $x^i, x_2 \in X_i^{\text{env}}$,

$$f_j(x^i) \neq f_j(x_2). \tag{53}$$

By (47), (49), (52), and (53),

$$f_j(x^j) < f_j(x_2) < f_j(x^i). \tag{54}$$

From (46), $x^i, x^j \in X_E$. Therefore, it follows from (48), (51), and (54) that Y_{iWE} is disconnected.

$(\Leftarrow)$ Assume that Y_{iWE} is disconnected. Then by Definition 5.2, there exist $x_1, x_2 \in X_{WE}$ and $\bar{x} \in X_i^{\text{env}}$ such that

$$\bar{x} \notin X_{WE}, \text{ and} \tag{55}$$

$$f_j(x_1) < f_j(\bar{x}) < f_j(x_2). \tag{56}$$

Since $x_1 \in X_{WE}$, (55) and (56) imply that

$$f_i(x_1) < f_i(\bar{x}). \tag{57}$$

Let $\hat{x} :=$ arg min$\{f_i(x) : f_j(x) = f_j(x_1), x \in X\}$. Then

$$f_j(\hat{x}) = f_j(x_1), \tag{58}$$

$$f_i(\hat{x}) \leq f_i(x_1), \text{ and} \tag{59}$$

$$\hat{x} \in X_i^{\text{env}}. \tag{60}$$

(57) and (59) imply that

$$f_i(\hat{x}) < f_i(\bar{x}), \tag{61}$$

and (56) and (58) imply that

$$f_j(\hat{x}) < f_j(\bar{x}). \tag{62}$$

Since $\bar{x} \in X_i^{\text{env}}$, (60), (61), and (62) imply that an augmented duality gap exists for $\varepsilon = \bar{y} := f_j(\bar{x})$. ■

THEOREM 5.2. *Suppose that there exists a neighborhood V of $y = \varepsilon$ such that env$_i(y)$ is continuous and differentiable over V. If Y_{iWE} is connected, then the optimal value of $P_i(\varepsilon)$ equals the optimal value of $QLD_i(\varepsilon)$ for all $\varepsilon \leq y^i$, where $y^i := f_j(x^i), j \neq i$, and x^i is defined as in (46).*

Proof. (by contradiction). Assume that there exists $\tilde{\varepsilon} \leq y^i$ such that

$$f_i(x^*) < \Theta_i(a^*, b^*, \tilde{\varepsilon}), \tag{63}$$

where x^* solves $P_i(\tilde{\varepsilon})$, and (a^*, b^*) solves $QLD_i(\tilde{\varepsilon})$. Let

$$\tilde{x} := \text{arg min}\{f_i(x) : f_j(x) = \tilde{\varepsilon}, j \neq i, x \in X\}. \tag{64}$$

Then

$$\tilde{x} \in X_i^{\text{env}}, \text{ and} \tag{65}$$

$$f_j(\tilde{x}) \leq f_j(x^i). \tag{66}$$

In addition, from (31), (32) and (64) it follows that

$$\begin{aligned} \Theta_i(a^*, b^*, \tilde{\varepsilon}) &:= \min\{QL_i(x, a^*, b^*, \tilde{\varepsilon}) : x \in X\} \\ &\leq QL_i(\tilde{x}, a^*, b^*, \tilde{\varepsilon}) \\ &= f_i(\tilde{x}) + a^*(f_j(\tilde{x}) - \tilde{\varepsilon})^2 + b^*(f_j(\tilde{x}) - \tilde{\varepsilon}) \\ &= f_i(\tilde{x}). \end{aligned} \tag{67}$$

Inequalities (63) and (67) imply that

$$f_i(x^*) < f_i(\tilde{x}). \tag{68}$$

Since x^* solves $P_i(\tilde{\varepsilon})$,

$$x^* \in X_{WE}, \tag{69}$$

$$f_j(x^*) \leq f_j(\tilde{x}). \tag{70}$$

From (64), it cannot happen that $f_j(x^*) = f_j(\tilde{x})$ and $f_i(x^*) < f_i(\tilde{x})$. Hence, (68) and (70) imply that

$$f_j(x^*) < f_j(\tilde{x}). \tag{71}$$

Inequalities (68) and (71) imply that

$$\tilde{x} \notin X_{WE}. \tag{72}$$

However, by (46),

$$x^i \in X_{WE}. \tag{73}$$

From (64), (66), (72), and (73) it follows that

$$f_j(\tilde{x}) < f_j(x^i). \tag{74}$$

Statements (65), (69), (71), (72), (73), and (74) imply that Y_{iWE} is disconnected. Therefore, if Y_{iWE} is connected,

$$f_i(x^*) \geq \Theta_i(a^*, b^*, \varepsilon) \text{ for all } \varepsilon \leq y^i. \tag{75}$$

However, Theorem 4.2 proves that

$$f_i(x^*) \leq \Theta_i(a^*, b^*, \varepsilon) \text{ for all feasible } \varepsilon. \tag{76}$$

Inequalities (75) and (76) imply that $f_i(x^*) = \Theta_i(a^*, b^*, \varepsilon)$ for all $\varepsilon \leq y^i$ if Y_{iWE} is connected. ■

Theorems 5.1 and 5.2 result in the following corollary which provides a simple tool for detecting the existence of an augmented duality gap and the disconnectedness of Y_{iWE}.

COROLLARY 5.1. *Let $\varepsilon \leq f_j(x^i)$, where x^i is defined as in (46). Suppose that there exists a neighborhood V of $y = \varepsilon$ such that env$_i(y)$ is continuous and differentiable over V. If the optimal value of $P_i(\varepsilon)$ is less than the optimal value of $QLD_i(\varepsilon)$, then there is an augmented duality gap for $\varepsilon = y^i = f_j(x^i)$.*

We now introduce a second quadratic Lagrangian dual problem which is an even more powerful tool for determining the disconnectedness of Y_{iWE} over a certain range of values of ε.

Let x^* solve $P_i(\varepsilon)$ and $\varepsilon^* := f_j(x^*), j \neq i$. The augmented quadratic Lagrangian dual problem, $AQLD_i(\varepsilon)$, is given as

$$\begin{aligned} AQLD_i(\varepsilon): \ & \text{maximize} && \Psi_i(a, b, \varepsilon) \\ & \text{subject to} && \varepsilon^* \leq -b/(2a) + \varepsilon \leq \varepsilon \\ & && a > 0 \\ & && b \text{ free}, \end{aligned} \tag{77}$$

where

$$\Psi_i(a, b, \varepsilon) := b^2/(4a) + \Theta_i(a, b, \varepsilon), \tag{78}$$

and $\Theta_i(a, b, \varepsilon)$ is defined as in (32).

Solving $AQLD_i(\varepsilon)$ is equivalent to finding the quadratic function supporting the set Y_i whose vertex has the z-co-ordinate maximized and the y-co-ordinate between the fixed scalars ε^* and ε. When $\varepsilon^* = \varepsilon$, the problem $AQLD_i(\varepsilon)$ is trivial. Therefore, we are primarily interested in the cases where ε is chosen such that $\varepsilon^* < \varepsilon$.

Theorem 5.3 below provides an alternate means for determining whether an augmented duality gap exists over a range of ε values. It also shows the relationship between the optimal values of the primal problem $P_i(\varepsilon)$ and the dual problem $AQLD_i(\varepsilon)$ if indeed an augmented duality gap does exist.

THEOREM 5.3. *Let x^i be defined as in (46). Let x^* solve $P_i(\varepsilon)$ with $\varepsilon \leq f_j(x^i), j \neq i$, and (a^*, b^*) solve $AQLD_i(\varepsilon)$. If $f_i(x^*) < \Psi_i(a^*, b^*, \varepsilon)$, then any duality gap which exists is augmented.*

Proof. Since x^* solves $P_i(\varepsilon)$, $f_i(x^*) \leq f_i(x)$ for all $x \in X$ such that $f_j(x) = f_j(x^*)$. It follows from Definition 2.4 that

$$x^* \in X_i^{\text{env}}. \tag{79}$$

Since (a^*, b^*) solves $AQLD_i(\varepsilon)$,

$$\varepsilon^* \leq -b^*/(2a^*) + \varepsilon \leq \varepsilon, \text{ and} \tag{80}$$

there exists

$$\bar{x} \in X_i^{\text{env}} \tag{81}$$

such that

$$f_j(\bar{x}) = -b^*/(2a^*) + \varepsilon. \tag{82}$$

From (80), (82), and the definition of ε^*, we get that

$$f_j(x^*) \leq f_j(\bar{x}) \leq \varepsilon. \tag{83}$$

If $f_i(x^*) < \Psi_i(a^*, b^*, \varepsilon)$, then

$$\begin{aligned} f_i(x^*) &< b^{*2}/(4a^*) + \Theta_i(a^*, b^*, \varepsilon) && (84)\\ &= b^{*2}/(4a^*) + \min\{QL_i(x, a^*, b^*, \varepsilon) : x \in X\} && (85)\\ &\leq b^{*2}/(4a^*) + QL_i(\bar{x}, a^*, b^*, \varepsilon) \\ &= b^{*2}/(4a^*) + f_i(\bar{x}) + a^*(f_j(\bar{x}) - \varepsilon)^2 + b^*(f_j(\bar{x}) - \varepsilon) && (86)\\ &= b^{*2}/(4a^*) + f_i(\bar{x}) + a^*(-b^*/(2a^*))^2 + b^*(-b^*/(2a^*)) && (87)\\ &= f_i(\bar{x}), && (88) \end{aligned}$$

where (84) follows from (78), (85) follows from (32), (86) follows from (31), and (87) follows from (82).

From (81) and (88),

$$f_j(x^*) \neq f_j(\bar{x}). \quad (89)$$

Inequalities (83) and (89) imply that

$$f_j(x^*) < f_j(\bar{x}) \leq \varepsilon. \quad (90)$$

Hence, if a duality gap exists, statements (79), (81), (88), and (90) show that it is augmented. ∎

The difference $\Psi_i(a^*, b^*, \varepsilon) - f_i(x^*)$ can be considered as the depth of the augmented duality gap.

Like the quadratic Lagrangian dual problem, the augmented quadratic Lagrangian dual problem seeks quadratic functions which support the image set Y_i. The greatest difference between these two dual problems is the location of the point of support relative to the set Y_i. For $QLD_i(\varepsilon)$, the point of support $(\hat{y}, \hat{z})$ corresponding to the optimal solution $\hat{x}$ is such that $\hat{y} = f_j(\hat{x}) = \varepsilon$, whereas for $AQLD_i(\varepsilon)$ the point of support $(\bar{y}, \bar{z})$ is such that $\bar{y} \leq \varepsilon$.

6. Example

This section contains an example of a nonconvex bicriteria programming problem which demonstrates some of the results of Sections 4 and 5. Consider the problem

$$\begin{aligned} BCP: \quad & \text{minimize } \{x^3 - 9x^2 + 26x - 22, 2 - x\} \\ & \text{subject to } 1 \leq x \leq 5. \end{aligned}$$

In the outcome space, $\text{env}_1(y) = -y^3 - 3y^2 - 2y + 2$ with $-3 \leq y \leq 1$. The set of outcomes Y_1 for this problem is shown in Figure 1 with the nondominated set highlighted. It can be easily seen that Y_{1E} (and also Y_{1WE}) is disconnected.

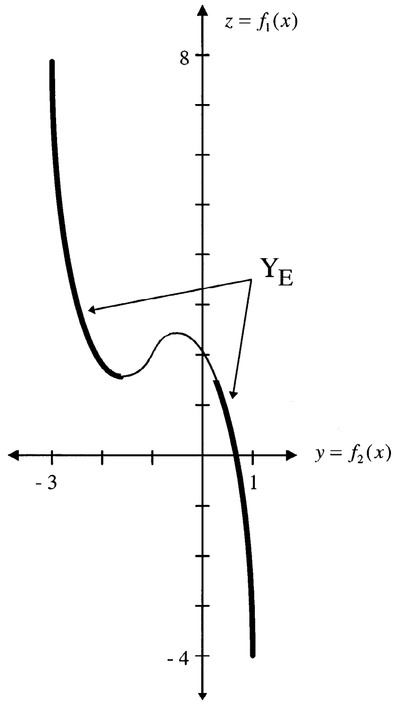

Figure 1. The set of outcomes and the nondominated set.

The ε-constraint problem formulated by retaining the function $f_1(x) = x^3 - 9x^2 + 26x - 22$ as the objective function and transforming the function $f_2(x) = 2 - x$ into a constraint is

$$P_1(\varepsilon): \quad \begin{array}{ll} \text{minimize} & x^3 - 9x^2 + 26x - 22 \\ \text{subject to} & 2 - x \le \varepsilon \\ & 1 \le x \le 5. \end{array}$$

It can be shown that a duality gap exists between the problem $P_1(\varepsilon)$ and the corresponding dual problem $LD_1(\varepsilon)$ for any value of ε such that $-2 < \varepsilon < 1$. For example, with $\varepsilon = 0$, the optimal value of $P_1(0)$ is $(18 - 2\sqrt{3})/9$, which corresponds to an optimal solution of $x^* = (9 + \sqrt{3})/3$, and the optimal value of the dual problem is -2. Note also that alternate optimal solutions exist in the outcome space for $P_1(\varepsilon)$ with $\varepsilon = (2\sqrt{3} - 3)/3$. This is all depicted in Figure 2.

Moreover, for any value of ε such that $(-3 - \sqrt{3})/3 < \varepsilon < 1$, the duality gap is augmented. Again, consider $\varepsilon = 0$. For $x_1 = (9 + \sqrt{3})/3$ and $x_2 = (9 - \sqrt{3})/3$ we get $f_1(x_1) = (18 - 2\sqrt{3})/9$, $f_1(x_2) = (18 + 2\sqrt{3})/9$, $f_2(x_1) = (-3 - \sqrt{3})/3$, and $f_2(x_2) = (-3 + \sqrt{3})/3$. This augmented gap can also be seen in Figure 2. Together, Figures 1 and 2 provide a visual representation of Theorem 5.1.

Now consider $QLD_1(\varepsilon)$. For any value of ε such that $(-3 - \sqrt{3})/3 < \varepsilon < (2\sqrt{3} - 3)/3$, the optimal value of $QLD_1(\varepsilon)$ is greater than the optimal value

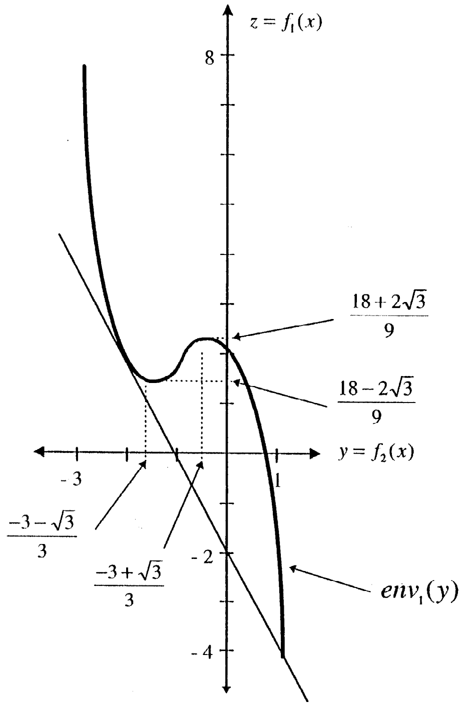

Figure 2. The augmented duality gap.

of $P_1(\varepsilon)$. For example, with $\varepsilon = 0$ the optimal value of $QLD_1(0)$ is 2 (with $a^* \geq 4, b^* = 2$, and $c^* = 2$), which is greater than the optimal value of $P_1(0)$ given in the preceding paragraph. In contrast to $P_1(\varepsilon)$, note that $QLD_1(\varepsilon)$ delivers a unique optimal solution in the outcome space with $\varepsilon = (2\sqrt{3} - 3)/3$. These results relate to Theorem 4.2, Theorem 5.2, and Corollary 5.1, and are depicted in Figure 3.

Finally consider $AQLD_1(\varepsilon)$. For any value of ε such that $(-3 - \sqrt{3})/3 < \varepsilon < (2\sqrt{3} - 3)/3$, the optimal value of $AQLD_1(\varepsilon)$ is greater than the optimal value of $P_1(\varepsilon)$. Again, consider $\varepsilon = 0$. Then $\varepsilon^* = (-3 - \sqrt{3})/3$, and the optimal value of $AQLD_1(0)$ is $(18 + 2\sqrt{3})/9$ (with $a^* = 4, b^* = (24 - 8\sqrt{3})/3$, and $c^* = (-30 + 26\sqrt{3})/9)$, which again, is greater than the optimal value of $P_1(0)$ given above. This result is related to Theorem 5.3, and is depicted in Figure 4.

In order to summarize the effectiveness of each of the problems $P_1(\varepsilon), QLD_1(\varepsilon)$, and $AQLD_1(\varepsilon)$ for this example, consider the following subdivision of the interval $[-3, 1]$ into four separate ranges. Let $R_1 := [-3, (-3 - \sqrt{3})/3]$, $R_2 := ((-3 - \sqrt{3})/3, (-3 + \sqrt{3})/3]$, $R_3 := ((-3 + \sqrt{3})/3, (2\sqrt{3} - 3)/3]$, and $R_4 := ((2\sqrt{3} - 3)/3, 1]$.

For each value of $\varepsilon \in (R_1 \cup R_4), P_1(\varepsilon)$ will generate an optimal solution x^* such that $f_2(x^*) = \varepsilon$, and the value of $\text{env}_i(y)|_{y=\varepsilon}$ is found. However, for all values of $\varepsilon \in (R_2 \cup R_3)$, except possibly $(2\sqrt{3} - 3)/3$, $P_1(\varepsilon)$ will generate a solution x^*

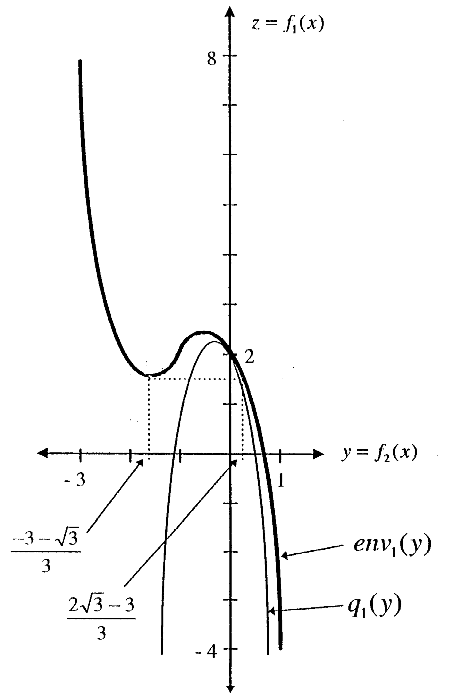

Figure 3. The optimal solution of $QLD_1(0)$.

such that $f_2(x^*) = (-3 - \sqrt{3})/3$. Therefore, for this example, $P_1(\varepsilon)$ gives very limited information about the behavior of $\mathrm{env}_1(y)$ over $(R_2 \cup R_3)$ and is unable to generate solutions whose images are in this interval.

Although it is a more difficult problem to solve, $QLD_1(\varepsilon)$ will generate the same optimal solution as $P_1(\varepsilon)$, denoted now by $\hat{x}$, for each value of $\varepsilon \in (R_1 \cup R_4)$, and thus, $f_2(\hat{x}) = \varepsilon$. Moreover, for each value of $\varepsilon \in (R_2 \cup R_3)$, $QLD_1(\varepsilon)$ will still generate an optimal solution $\hat{x}$ such that $f_2(\hat{x}) = \varepsilon$. $QLD_1(\varepsilon)$ eliminates ambiguity related to alternate optimal solutions, enables us to find the value of $\mathrm{env}_1(y)$ for all feasible values of $y = f_2(x)$, and can generate all solutions, even those in the gap. From this information it is easy to determine whether an augmented duality gap exists and if the set of weakly nondominated solutions is disconnected.

For $AQLD_1(\varepsilon)$, the most interesting results of this example are for $\varepsilon \in R_3$. For any value of $\varepsilon \in \mathrm{int}(R_3)$, $AQLD_1(\varepsilon)$ will generate a solution $\bar{x}$ such that $f_2(\bar{x}) = (-3 + \sqrt{3})/3$, for $a > 0$ sufficiently large. In accordance with Definition 5.1 or Theorem 5.3, the values $f_1(x^*), f_1(\bar{x}), f_2(x^*)$, and $f_2(\bar{x})$, where x^* solves $P_1(\varepsilon)$ and $\bar{x}$ solves $AQLD_1(\varepsilon)$, are evidence of an augmented duality gap and the disconnectedness of Y_{iWE}. Hence, for this entire range of values of ε, the optimal value of $AQLD_1(\varepsilon)$ remains fixed and delivers information about the depth of the augmented gap.

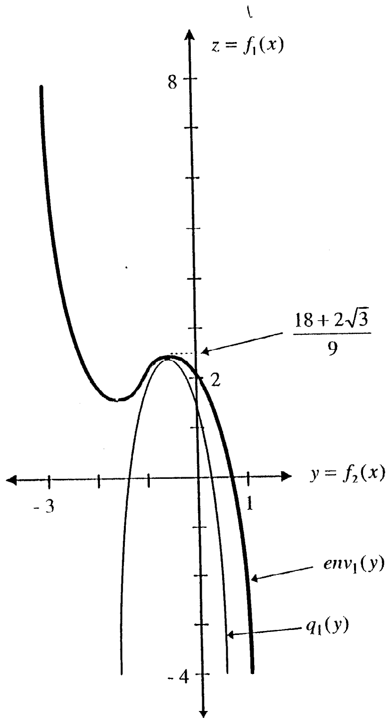

Figure 4. The optimal solution of $AQLD_1(0)$.

7. Conclusions

This paper is a continuation of the research done on applying generalized Lagrangian duality to multiple objective programming and presents a methodology specially tailored for bicriteria nonconvex programs. Selecting a specific form of the generalized Lagrange function leads to the concept of q_i-approachability and the development of two quadratic Lagrangian dual problems.

The concept of the lower envelope function is employed since the information about local values of this function (and its derivative) comes from solving a related single objective program. The lower envelope function helps relate q_i-approachability to efficiency.

Extensive consideration is given to the theoretical relationships between the two quadratic dual problems and efficient solutions. These problems also provide a means of determining whether or not the set of weakly nondominated solutions is connected. The structure of this set motivates the introduction of the concept of an augmented duality gap. Existence of the gap, defined as being related to disconnectedness of the set of weakly nondominated solutions, can be detected locally at a given point or over a certain range. The enclosed example illustrates both cases.

The results presented in this paper can be extended in several directions. First, they can be generalized for the multiple objective case, which would require many modifications due to the multidimensionality of the image space. Also, certain concepts, like $\mathrm{env}_i(y)$, would have to be redefined. Second, the means of detecting the augmented duality gap could be applied to single objective programs in order to better explore the lack of convexity in general. Finally, one may develop other nonlinear Lagrange functions and examine their capability of dealing with multiple objective programs.

The authors believe that research on multiple objective programming and generalized Lagrangian duality deserves further studies since both fields can benefit substantially from each other.

References

Aksoy, Y., An Interactive Branch and Bound Algorithm for Bicriterion Nonconvex/Mixed Integer Programming, *Naval Research Logistics* **37**, pp. 403–417, 1990.

Ashton, D.J. and D.R. Atkins, Multicriteria Programming for Financial Planning: Some Second Thoughts, in *Multiple Criteria Analysis – Operational Methods*, Peter Nijkamp and Jaap Spronk (eds.), Gower Publishing Company Limited, Aldershot, Hampshire, England, pp. 11–23, 1981.

Bazaraa, M.S., Geometry and Resolution of Duality Gaps, *Naval Research Logistics Quarterly* **20**, pp. 357–366, 1973.

Benson, H.P., Existence of Efficient Solutions for Vector Maximization Problems, *Journal of Optimization Theory and Applications* **26**, pp. 569–580, 1978.

Benson, H.P., Vector Maximization with Two Objective Functions, *Journal of Optimization Theory and Applications* **28**, pp. 253–257, 1979.

Benson, H.P. and T.L. Morin, The Vector Maximization Problem: Proper Efficiency and Stability, *SIAM Journal on Applied Mathematics* **32**, pp. 64–72, 1977.

Benson, H.P, and T.L. Morin, Bicriteria Mathematical Programming Model for Nutrition Planning in Developing Nations, *Management Science* **33**, pp. 11593–11601, 1987.

Bernau, H., Use of Exact Penalty Functions to Determine Efficient Decisions, *European Journal of Operational Research* **49**, pp. 348–355, 1990.

Cambini, A., and L. Martein, Linear Fractional and Bicriteria Linear Fractional Programs, *Generalized Convexity and Fractional Programming with Economic Application*, A. Cambini, E. Castagnoli, L. Martein, P. Mazzoleni, and S. Schaible (eds.), Springer-Verlag, Berlin, pp. 155–166, 1988.

Choo, E.U. and D.R. Atkins, Bicriteria Linear Fractional Programming, *Journal of Optimization Theory and Applications* **36**, pp. 203–220, 1982.

Eschenauer, H., J. Koski, and A. Osyczka, *Multicriteria Design Optimization—Procedures and Applications*, Springer-Verlag, Berlin, 1990.

Gaughan, E.D., *Introduction to Analysis*, Wadsworth Publishing Company, Inc., 1975.

Gearhart, W.B., Technical Note: On the Characterization of Pareto-Optimal Solutions in Bicriterion Optimization, *Journal of Optimization Theory and Applications* **7**, pp. 301–307, 1979.

Geoffrion, A.M., Solving Bicriterion Mathematical Programs, *Operations Research* **15**, pp. 39–54, 1967.

Gould, F.J., Extensions of Lagrange Multipliers in Nonlinear Programming, *SIAM Journal of Applied Mathematics* **17**, pp. 1280–1297, 1969.

Haimes, Y.Y. and V. Chankong, *Multiobjective Decision Making—Theory and Methodology*, North Holland, New York, 1983.

Helbig, S., On a Constructive Approximation of the Efficient Outcomes in Bicriterion Vector Optimization, *Journal of Optimization Theory and Applications* **78**, pp. 613–622, 1993.

Jahn, J. and A. Merkel, Reference Point Approximation Method for the Solution of Bicriteria Nonlinear Optimization Problems, *Journal of Optimization Theory and Applications* **4**, pp. 87–103, 1992.

Jahn, J., J. Klose and A. Merkel, On the Application of a Method of Reference Point Approximation to Bicriterial Optimization Problems in Chemical Engineering, *Advances in Optimization*, W. Oettli and D. Pallaschke (eds.), Springer-Verlag, Berlin, pp. 478–491, 1992.

Jueschke, A., J. Jahn and A. Kirsch, A Bicriterial Optimization Problem of Antenna Design, to appear in *Computational Optimization and Applications*.

Kaliszewski, I., Norm Scalarization and Proper Efficiency in Vector Optimization, *Foundations of Control Engineering* **11**, pp. 117–131, 1986.

Kopsidas, G.C., Multiobjective Optimization of Table Olive Preparation Systems, *European Journal of Operations Research* **85**, pp. 383–398, 1995.

Kostreva, M.M., T. Ordoyne, and M. Wiecek, Multiple-objective Programming with Polynomial Objectives and Constraints, *European Journal of Operational Research* **57**, pp. 381–394, 1992.

Ku Chi-Fa, Some Applications of Multiple Objective Decision Making, in *Operational Research '81*, J.P. Brans (ed.), North-Holland, Amsterdam, pp. 395–404, 1981.

Lee, S.M. and A.J. Wynne, Separable Goal Programming, in *Multiple Criteria Analysis–Operational Methods*, Peter Nijkamp and Jaap Spronk (eds.), Gower Publishing Company Limited, Aldershot, Hampshire, England, pp. 117–136, 1981.

Luc, D.T., Connectedness of the Efficient Point Sets in Quasiconcave Vector Maximization, *Journal of Mathematical Analysis and Applications* **122**, pp. 346–354, 1987.

Martein, L., On the Bicriteria Maximization Problem, *Generalized Convexity and Fractional Programming with Economic Application*, A. Cambini, E. Castagnoli, L. Martein, P. Mazzoleni, and S. Schaible (eds.), Springer-Verlag, Berlin, pp. 14–22, 1988.

Minoux, M., *Mathematical Programming – Theory and Algorithms*, John Wiley, Chichester, 1986.

Naccache, P.H., Connectedness of the Set of Nondominated Outcomes in Multicriteria Optimization, *Journal of Optimization Theory and Applications* **25**, pp. 459–467, 1978.

Nakayama, H., H. Sayama, and Y. Sawaragi, A Generalized Lagrangian Function and Multiplier Method, *Journal of Optimization Theory and Aplications* **17**, pp. 211–227, 1975.

Osyczka, A., *Multicriterion Optimization in Engineering with Fortran Programs*, Ellis Horwood, Chichester, 1984.

Osyczka, A. and J. Zajac, Multicriteria Optimization of Computationally Expensive Functions and its Application to Robot Spring Balancing Mechanism Design, *Multicriteria Design Optimization–Procedures and Applications*, H.A. Eschenauer, J. Koski, and A. Osyczka (eds.), Springer-Verlag, Berlin, pp. 168–183, 1990.

Pareto, V., *Cours d'Economie Politique*, Rouge, Lausanne, Switzerland, 1896.

Payne, H.J., E. Polak, D.C. Collins, and W.S. Meisel, An Algorithm for Bicriteria Optimization Based on the Sensitivity Function, *IEEE Transactions on Automatic Control*, AC-20, pp. 546–548, 1975.

Payne A.N., and E. Polak, An Interactive Rectangle Elimination Method for Bi-Objective Decision Making, *IEEE Transactions on Automatic Control*, AC-25, pp. 421–432, 1980.

Payne, A.N., Efficient Approximate Representation of Bi-Objective Tradeoff Sets, *Journal of the Franklin Institute* **330**, pp. 1219–1233, 1993.

Rietveld, P., *Multiple Objective Decision Methods and Regional Planning*, North Holland, Amsterdam, 1980.

Rockafellar, R.T., Augmented Lagrange Multiplier Functions and Duality in Nonconvex Programming, *SIAM Journal on Control* **12**, pp. 268–285, 1974.

Roy, A. and J. Wallenius, Nonlinear Multiple Objective Optimization: An Algorithm and Some Theory, Faculty working paper DIS 85–86, College of Business, Arizona State University, Tempe, AZ, 1988.

Schaible, S., Bicriteria Quasiconcave Programs, *Cahiers du Centre d'Etudes de Recherche Operationnelle* **25**, pp. 93–101, 1983.

Steuer, R.E. and E.U. Choo, An Interactive Weighted Tchebycheff Procedure for Multiple Objective Programming, *Mathematical Programming* **26** pp. 309–322, 1983.

Tabek, D, A.A. Schy, D.P. Giesy and K.G. Johnson, Application of Multiobjective Optimization in Aircraft Control Systems Design. *Automatica* **15**, pp. 595–600, 1979.

TenHuisen, M.L., Generalized Lagrangian Duality in Multiple Objective Programming, Ph.D. thesis, Clemson University, Clemson, S.C., 1993.

TenHuisen, M.L. and M.M. Wiecek, A New Approach to Solving Multicriteria Nonconvex Optimization Problems, *Support Systems for Decision and Negotiation Processes*, R. Kulikowski, Z. Nahorski, J.W. Owsinski, and A. Straszak (eds.), System Research Institute Polish Academy of Sciences, Warsaw, 1992.

TenHuisen, M.L. and M.M. Wiecek, Vector Optimization and Generalized Lagrangian Duality, *Annals of Operations Research* **51**, pp. 15–32, 1994.

TenHuisen, M.L. and M.M. Wiecek, On the Structure of the Nondominated Set for Bicriteria Programs, to appear in *Multi-Criteria Decision Analysis*, 1996.

Tind, J. and L.A. Wolsey, An Elemementary Survey of General Duality Theory in Mathematical Programming, *Mathematical Programming* **21**, pp. 241–261, 1981.

Walker, J., An Interactive Method As an Aid in Solving Bicriterion Mathematical Programming Problems, *Journal of the Operational Research Society* **29**, pp. 915–922, 1978.

Warburton, A.R., Quasiconcave Vector Maximization: Connectedness of the Sets of Pareto-optimal and Weak Pareto-optimal Alternatives, *Journal of Optimization Theory and Applications* **40**, pp. 537–557, 1983.

Warburton, A.R. Parametric Solution of Bicriterion Linear Fractional Programs, *Operations Research* **33**, pp. 74–84, 1985.

Wendell, R.E., and D.N. Lee, Efficiency in Multiple Objective Optimization Problems, *Mathematical Programming* **12**, pp. 406–414, 1977.

Wierzbicki, A.P., The Use of Reference Objectives in Multiobjective Optimization, *Multiple Criteria Decision Making Theory and Applications*, G. Fandel and T. Gal (eds.), Springer Verlag, Berlin, pp. 468–486, 1980.

Wierzbicki, A.P., On the Completeness and Constructiveness of Parametric Characterizations to Vector Optimization Problems, *OR-Spektrum* **8**, pp. 73–87, 1986.